SADLER UNIT **4** MATHEMATICS METHODS

WORKED SOLUTIONS

Chapter 3 Continuous random variables

Exercise 3A

Question 1

a $\frac{163}{186}$

b $\frac{128}{186} = \frac{64}{93}$

c $\frac{4}{186} = \frac{2}{93}$

Question 2

a $\frac{3+7+14}{67} = \frac{24}{64} \sim 36\%$

b $\frac{3+7}{67} = \frac{10}{67} \sim 15\%$

c $\frac{5+3+3+4+1}{67} = \frac{16}{67} \sim 24\%$

d $\frac{17+10+5}{67} = \frac{32}{67} \sim 48\%$

e $\frac{5+3+3+4+1}{67} = \frac{16}{67} \sim 24\%$

f $\frac{17+10}{3+7+14+17+10} = \frac{27}{51} \sim 53\%$

Question 3

a $0.4 \times 50 = 20$

b **i** $0.38 + 0.28 + 0.12 + 0.04 = 0.82$

ii $0.02 + 0.16 = 0.18$

iii $\dfrac{0.16}{0.16 + 0.38 + 0.28} = \dfrac{0.16}{0.82} = \dfrac{8}{41}$

Question 4

a $0.325 + 0.2 + 0.075 + 0.025 + 0.025 = 0.65$

b 0.65

c $0.075 + 0.075 + 0.15 + 0.325 + 0.2 = 0.825$

d $\dfrac{0.025 + 0.025 + 0.075}{0.65} = \dfrac{0.125}{0.650} = \dfrac{5}{26}$

e $\dfrac{0.6}{0.75} = 0.8$

Question 5

54 apples in sample.

a $\dfrac{4 + 11}{54} = \dfrac{15}{54} \approx 0.28$

b $1 - 0.28 = 0.72$

c $\dfrac{15 + 10 + 7 + 4}{54} = \dfrac{36}{54} \approx 0.67$

d $\dfrac{15 + 10}{39} = \dfrac{25}{39} \approx 0.64$

Exercise 3B

Question 1

$$\begin{aligned} 4k &= 1 \\ k &= 0.25 \end{aligned}$$

Question 2

$$\begin{aligned} 20k &= 1 \\ k &= 0.05 \end{aligned}$$

Question 3

$$\begin{aligned} 0.5k &= 1 \\ k &= 2 \end{aligned}$$

Question 4

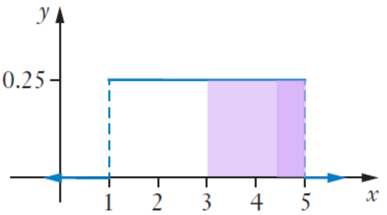
$$\begin{aligned} (k-1) \times 0.5 &= 0.75 \\ k-1 &= 1.5 \\ k &= 2.5 \end{aligned}$$

Question 5

$$\begin{aligned} (k-1) \times 1 &= 0.8 \\ k &= 1.8 \end{aligned}$$

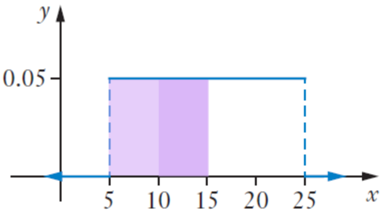
Question 6

$$\begin{aligned} \frac{\mathrm{P}(X > (k-1))}{0.5} &= 0.25 \\ \mathrm{P}(X > k-1) &= 0.125 \\ \mathrm{P}(X < k-1) &= 0.875 \\ k-1 &= 0.875 \times 4 \\ &= 3.5 \\ k &= 4.5 \end{aligned}$$



$\mathrm{P}(X > k \mid X > 3) = 0.25$

Question 7



$P(X > 10 \mid X < k) = 0.5$

$$\frac{P(10 < X < k)}{P(X < k)} = 0.5$$

If $P(10 < X < k) = 0.5 \times P(5 < X < k)$,

$$k = 15$$

or

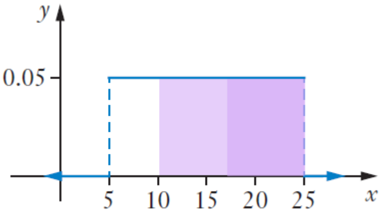
$$(k - 10) \times 0.05 = \frac{1}{2}(k - 5) \times 0.05$$

$$k - 10 = \frac{1}{2}(k - 5)$$

$$2k - 20 = k - 5$$

$$k = 15$$

Question 8



$P(X > k \mid X > 10) = 0.5$

$$\frac{P(X > k)}{P(X > 10)} = 0.5$$

$$P(X > k) = 0.5 \times 0.75$$

$$= 0.375$$

$$\frac{3}{8} \times 20 = 7.5$$

$$25 - 7.5 = 17.5$$

Question 9

$$2k = 1$$

$$k = 0.5$$

$$f(x) = \begin{cases} 0.5 & \text{for } 1 \le x \le 3 \\ 0 & \text{otherwise} \end{cases}$$

Question 10

a $\mathrm{P}(X < 4) = \frac{1}{4}$

b $\mathrm{P}(X = 4) = 0$

c $\mathrm{P}(X < 8) = \frac{3}{4}$

d

$$\begin{aligned}&\mathrm{P}(X > 4 \mid X < 8)\\&= \frac{\mathrm{P}(4 < X < 8)}{\mathrm{P}(X < 8)}\\&= \frac{0.5}{0.75}\\&= \frac{2}{3}\end{aligned}$$

Question 11

a

$$\begin{aligned}\mathrm{E}(X) &= \frac{a+b}{2}\\&= \frac{0.5+1.5}{2}\\&= 1\end{aligned}$$

b $\mathrm{P}(X > 1.2) = 0.3$

c $\mathrm{P}(X > 2) = 0$

d $\mathrm{P}(X < 2) = 1$

e

$$\begin{aligned}&\mathrm{P}(X < 1 \mid X < 1.3)\\&= \frac{\mathrm{P}(X < 1)}{\mathrm{P}(X < 1.3)}\\&= \frac{0.5}{0.8}\\&= 0.625\end{aligned}$$

Question 12

a $\mathrm{E}(X) = (0+50)\dfrac{1}{2}$

$= 25$

b 0

c $\mathrm{P}(X < 20) = \dfrac{2}{5} = 0.4$

d $\mathrm{P}(X \le 20) = \dfrac{2}{5} = 0.4$

e $\mathrm{P}(X < 20 \mid X < 25)$

$= \dfrac{\mathrm{P}(X < 20)}{\mathrm{P}(X < 25)}$

$= \dfrac{0.4}{0.5}$

$= 0.8$

f $\mathrm{P}(X < 25 \mid X < 20) = 1$

g $\mathrm{P}(X > 20 \mid X < 25)$

$= \dfrac{\mathrm{P}(20 \le X < 25)}{\mathrm{P}(X < 25)}$

$= \dfrac{0.1}{0.5}$

$= 0.2$

h $\mathrm{P}(X > 25 \mid X < 20) = 0$

Question 13

a $\mathrm{P}(X \le 20) = 0.5$

b $\mathrm{P}(X \ge 15) = \dfrac{25}{40} = 0.625$

c $\mathrm{P}(X \le 20 \mid X \ge 15)$

$= \dfrac{\mathrm{P}(15 \le X \le 20)}{\mathrm{P}(X \ge 15)}$

$= \dfrac{5}{25}$

$= \dfrac{1}{5}$

Exercise 3C

Question 1

$$\begin{aligned}\frac{1}{2}\times 4\times k &= 1\\ k &= 0.5\end{aligned}$$

Question 2

$$\begin{aligned}\frac{1}{2}\times 8\times k &= 1\\ k &= 0.25\end{aligned}$$

Question 3

$$\begin{aligned}\frac{1}{2}\times (0.6+1)\times k &= 1\\ k &= \frac{1}{0.8}\\ &= 1.25\end{aligned}$$

Question 4

$$\begin{aligned}\frac{1}{2}\times k\times 0.8 &= 1\\ k &= \frac{1}{0.4}\\ &= 2.5\end{aligned}$$

Question 5

$$\begin{aligned}3\times 4k+\frac{1}{2}\times 3\times 2k &= 1\\ 12k+3k &= 1\\ 15k &= 1\\ k &= \frac{1}{15}\end{aligned}$$

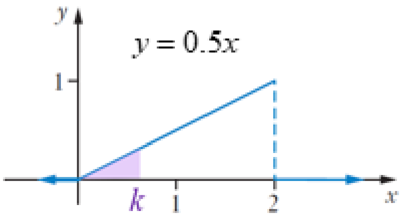
Question 6

$$\frac{1}{2} \times \pi \times k^2 = 1$$

$$k^2 = \frac{2}{\pi}$$

$$k = \sqrt{\frac{2}{\pi}}$$

Question 7



$$y = 0.5x$$

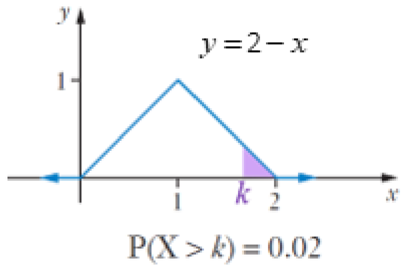
$$\frac{1}{2}k \times \frac{1}{2}k = 0.0625$$

$$\frac{1}{2}k^2 = 0.0625$$

$$k^2 = 0.25$$

$$k = 0.5$$

Question 8



$$P(X > 5) = 0.02 \quad \therefore k > 1$$

$$P(x < k) = 0.98$$

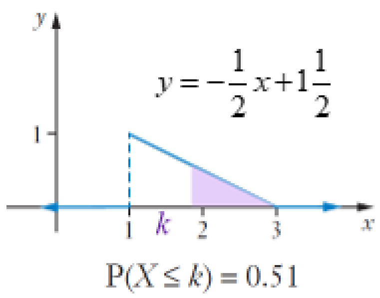
$$\frac{1}{2}(2-k)\times(2-k) = 0.02$$

$$(2-k)^2 = 0.04$$

$$2-k = 0.2$$

$$k = 1.8$$

Question 9



$$P(X \geq k) = 0.49$$

$$\frac{1}{2}(3-k)\times\left(-\frac{1}{2}k + 1\frac{1}{2}\right) = 0.49$$

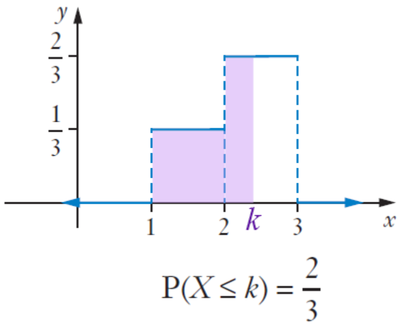
$$\frac{1}{2}\times\left(-\frac{1}{2}\right)(3-k)(k-3) = 0.49$$

$$(3-k)(k-3) = -1.96$$

$$k = 1.6,\ \cancel{4.4}$$

$$\therefore k = 1.6$$

Question 10



$$\frac{1}{3} + (k-2) \times \frac{2}{3} = \frac{2}{3}$$

$$(k-2) \times \frac{2}{3} = \frac{1}{3}$$

$$(k-2) = \frac{1}{2}$$

$$k = 2\frac{1}{2}$$

Question 11

$$\int_0^{0.5} kx^2 dx = 1$$

$$\left[\frac{kx^3}{3}\right]_0^{0.5} = 1$$

$$\frac{k}{3}\left(\frac{1}{2}\right)^3 - 0 = 1$$

$$\frac{k}{24} = 1$$

$$k = 24$$

Question 12

$$\int_1^2 kx^3 dx = 1$$

$$\left[\frac{kx^4}{4}\right]_1^2 = 1$$

$$\frac{16k}{4} - \frac{k}{4} = 1$$

$$\frac{15k}{4} = 1$$

$$k = \frac{4}{15}$$

Question 13

$$\int_0^{0.5} ke^x dx = 1$$

$$\left. ke^x \right]_0^{0.5} = 1$$

$$ke^{0.5} - ke^0 = 1$$

$$k(e^{0.5} - 1) = 1$$

$$k = \frac{1}{e^{0.5} - 1}$$

Question 14

$$\int_0^k \frac{x^2}{9}\, dx = 0.512$$

$$\left. \frac{x^3}{27} \right]_0^k = 0.512$$

$$\frac{k^3}{27} - \frac{0}{27} = 0.512$$

$$k^3 = 13.824$$

$$k = 2.4$$

Question 15

$$\int_0^k 1.5x^{\frac{1}{2}}\, dx = \frac{1}{8}$$

$$\frac{3}{2}\left[x^{\frac{3}{2}} \times \frac{2}{3} \right]_0^k = \frac{1}{8}$$

$$k^{\frac{3}{2}} - 0 = 2^{-3}$$

$$k = (2^{-3})^{\frac{2}{3}}$$

$$= 2^{-2}$$

$$= 0.25$$

Question 16

$$ke^{-2k} = ke^{-2x}$$

$$\therefore k = 2$$

Question 17

a $\frac{1}{2}\times 4\times k = 1$

$$k = 0.5$$

$$m = \frac{0.5}{4} = \frac{1}{8}$$

$$\therefore y = \frac{1}{8}x$$

$$f(x) = \begin{cases} \frac{1}{8}x & \text{for } 0 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

b $k + 2k + k = 1$

$$4k = 1$$

$$k = 0.25$$

$$f(x) = \begin{cases} 0.25 & \text{for } 1 \le x \le 2 \\ 0.5 & \text{for } 2 < x < 3 \\ 0.25 & \text{for } 3 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Question 18

$$f(x) = \begin{cases} 2x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

a $\frac{1}{2}\times 0.5\times 2(0.5) = 0.25$

b $\mathrm{P}(X < 0.75) = \frac{1}{2}\times 0.75\times 1.5$

$$= 0.5625$$

$$\mathrm{P}(X > 0.75) = 1 - 0.5625$$

$$= 0.4375$$

c $\mathrm{P}(X < 2) = 1$

d $\mathrm{P}(X > 0.5 \mid X < 0.75)$

$$= \frac{\mathrm{P}(0.5 < X < 0.75)}{\mathrm{P}(X < 0.75)}$$

$$= \frac{(0.5625 - 0.25)}{0.5625}$$

$$= \frac{5}{9}$$

Question 19

a $\mathrm{P}(X < 0.5) = 0.5 \times 1 = 0.5$

b Line involved: $m = -1$

$\therefore y = -x + 1.5$

When $x = 1, y = 0.5$

$$\mathrm{P}(X > 1) = \frac{1}{2} \times 0.5 \times 0.5$$

$$= \frac{1}{8}$$

c $\mathrm{P}(X < 1)$

$$= 1 - \frac{1}{8}$$

$$= \frac{7}{8}$$

d $\mathrm{P}(X > 0.5 \mid X < 1)$

$$= \frac{\mathrm{P}(0.5 < X < 1)}{\mathrm{P}(X < 1)}$$

$$= \frac{\left(\frac{7}{8} - \frac{1}{2}\right)}{\frac{7}{8}}$$

$$= \frac{3}{7}$$

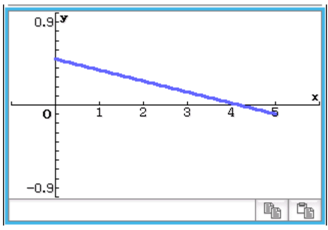
Question 20

$\int_0^5 (0.5 - 0.08x)\ dx$

$= 1.5 \quad \therefore$ not a pdf as the area is greater than 1

$\int_0^5 (0.5 - 0.12x)\ dx$

$= 1$



Intercepts : $(0,\ 0.5)\quad (4\frac{1}{6},\ 0)$.

The function has negative area and cannot be a pdf.

$\int_0^5 (0.5 - 0.08x)\ dx$

$= 1$

Intercepts : $(0,\ 0.4)$ and $(5,\ 0)$

This is a probability density function.

Question 21

a $\int_0^3 0.08(x+2)\,dx$

$= 0.08\int_0^3 (x+2)\,dx$

$= 0.08\left[\frac{x^2}{2} + 2x\right]_0^3$

$= 0.08(4.5 + 6 - 3)$

$= 0.84$

b $0.08\int_1^2 (x+2)\ dx = 0.28$

c $0.08\int_1^3 (x+2)\,dx = 0.64$

$\mathrm{P}(X \le 2 \mid X \ge 1)$

$= \frac{\mathrm{P}(1 \le X \le 2)}{\mathrm{P}(X \ge 1)}$

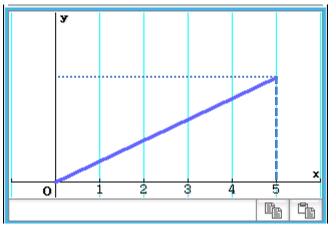
$= \frac{0.28}{0.64}$

$= \frac{7}{16}$

Question 22

a $\frac{1}{2}\times 5\times 5k = 1$

$k = 0.08$



b $P(X \le 4) = \frac{1}{2}\times 4\times(0.08\times 4)$

$= 0.64$

c $P(2 \le X \le 4) = \int_2^4 0.08x\,dx$

$= 0.48$

d $\frac{P(2 \le X \le 4)}{P(X < 4)} = \frac{0.48}{0.64}$

$= 0.75$

Question 23

a $\int_1^3 \frac{3}{2x^2}dx$

$= -\frac{3}{2}x^{-1}\Big]_1^3$

$= -\frac{3}{2}\times\frac{1}{3} - \left(-\frac{3}{2}\times 1\right)$

$= -\frac{1}{2} + \frac{3}{2}$

$= 1$

b $P(X \ge 2) = \int_2^3 \frac{3}{2x^2}dx$

$= 0.25$

c $P(2 \le X \le 2.5) = \int_2^{2.5} \frac{3}{2x^2}dx$

$= 0.15$

d $P(X \le 2.5 \mid X \ge 2)$

$= \frac{P(2 \le X \le 2.5)}{P(X \ge 2)}$

$= \frac{0.15}{0.25}$

$= 0.6$

Question 24

a $$\int_1^4 (x^2 + kx)dx = \left[\frac{x^3}{3} + \frac{kx^2}{2}\right]_1^4 = 1$$

$$\left(\frac{64}{3} + 8k\right) - \left(\frac{1}{3} + \frac{1}{2}k\right) = 1$$

$$21 + 7\frac{1}{2}k = 1$$

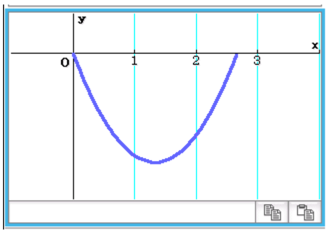
$$\frac{15}{2}k = -20$$

$$k = \frac{-20.2}{15}$$

$$= -\frac{8}{3}$$

b No, the graph of $y = x^2 - \frac{8x}{3}$

has negative values for $0 < x \le 2\frac{2}{3}$.



Question 25

a $k\int_1^3 (1-x)(x-3)dx = \frac{4}{3}k = 1$

$$k = \frac{3}{4}$$

b $\mathrm{P}(X \le 2) = \int_1^2 \frac{3}{4}(1-x)(x-3)dx$

$$= \frac{1}{2}$$

c $\mathrm{P}(X \le 2.5) = \int_1^{2.5} \frac{3}{4}(1-x)(x-3)dx$

$$= 0.84$$

d $\mathrm{P}(X \ge q) = 0.6$

$\therefore \mathrm{P}(X < q) = 0.4$

$$\int_1^q \frac{3}{4}(1-x)(x-3)dx$$

$$= \frac{3}{4}\int_1^q (1-x)(x-3)dx$$

$$= \frac{3}{4}\left[-\frac{q^3}{3} + 2q^2 - 3q + \frac{4}{3}\right]$$

$$\frac{3}{4}\left[-\frac{q^3}{3} + 2q^2 - 3q + \frac{4}{3}\right] = 0.4$$

$q = 0.339,\ 1.866$

0.339 is outside $1 \le x \le 3$

$\therefore q = 1.87$

Question 26

$$\int_1^4 \left(\frac{a+bx-x^2}{9}\right)dx = 1$$

$$\frac{1}{9}\left[ax+\frac{bx^2}{2}-\frac{x^3}{3}\right]_1^4 = 1$$

$$\left[ax+\frac{bx^2}{2}-\frac{x^3}{3}\right]_1^4 = 9$$

$$\left(4a+8b-\frac{64}{3}\right)-\left(a+\frac{1}{2}b-\frac{1}{3}\right)=9$$

$$3a-7\frac{1}{2}-21=9 \qquad \rightarrow \quad \text{Equation 1}$$

$$\int_1^2 \left(\frac{a+bx-x^2}{9}\right)dx = \frac{5}{27}$$

$$\left[ax+\frac{bx^2}{2}-\frac{x^3}{3}\right]_1^2 = \frac{5}{27}\times\frac{9}{1}$$

$$\left(2a+2b-\frac{8}{3}\right)-\left(a+\frac{1}{2}b-\frac{1}{3}\right)=\frac{5}{3}$$

$$a+\frac{3}{2}b-\frac{7}{3}=\frac{5}{3}$$

$$a+\frac{3}{2}b=4 \qquad \rightarrow \quad \text{Equation 2}$$

Solving simultaneously

$a=-5,\ b=6$

Question 27

a $f(3) = 0.4$

$$\begin{aligned} P(X < 3) &= 1 \times 2 \times 0.4 \\ &= 0.4 \end{aligned}$$

b $$\begin{aligned} P(X < 4 \mid X > 3) &= \frac{P(3 < X < 4)}{P(X > 3)} \\ &= \frac{0.4}{1 - P(X < 3)} \\ &= \frac{0.4}{0.6} \\ &= \frac{2}{3} \end{aligned}$$

Question 28

$$m_1 = \frac{1}{15} \times \frac{1}{15} = \frac{1}{225}$$

$$m_2 = -\frac{1}{15} \times \frac{1}{15} = -\frac{1}{225}$$

$$f(x) = \begin{cases} \dfrac{x-15}{225} & \text{for } 15 \le x \le 30 \\ \dfrac{45-x}{225} & \text{for } 30 \le x \le 45 \\ 0 & \text{otherwise} \end{cases}$$

a $\int_{15}^{25} \frac{x-15}{225}\,dx = \frac{2}{9}$

b $2\int_{25}^{30} \frac{x-15}{225}\,dx = \frac{5}{9}$

c $\int_{30}^{40} \frac{45-x}{225}\,dx = \frac{4}{9}$

$$\begin{aligned} &P(X < 40 \mid X > 30) \\ &= \frac{P(30 < X < 40)}{P(X > 30)} \\ &= \frac{4}{9} \div \frac{1}{2} \\ &= \frac{8}{9} \end{aligned}$$

Question 29

a $\int_{10}^{18} 0.025(x-10)\,dx = \dfrac{4}{5}$

b $\int_{14}^{18} 0.025(x-10)\,dx + \int_{18}^{20} 0.1(20-x)\,dx$

$= 0.6 + 0.2$

$= 0.8$

c $1 - \int_{19}^{20} 0.1(20-x)\,dx$

$= 1 - 0.05$

$= 0.95$

Question 30

a $f(x) = 0.2e^{-0.2x}$ or $\int_{0}^{\infty} 0.2e^{-0.2x}dx = 0.2019$

$\mathrm{P}(X > 8) = 1 - \mathrm{P}(X < 8)$

$\mathrm{P}(X < 8) = \int_{0}^{8} 0.2e^{-0.2x}dx$

$= -e^{-0.2x}\Big]_0^8$

$= -e^{-1.6} - \left(-e^{0}\right)$

$= 0.7981$

$1 - 0.7981 = 0.2019$

b $\mathrm{P}(\text{success}) = 0.2019$

$= \binom{6}{2}(0.2019)^2(0.7981)^4$

$= 0.2481$

Question 1

$\mathrm{E}(X)$ is midpoint as pdf is uniform $\Rightarrow \dfrac{7+2}{2} = 4.5$

Alternatively

$$\begin{aligned}\mathrm{E}(X) &= \int_2^7 0.2x\,dx \\ &= 4.5\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= \int_{-\infty}^{\infty} f(x)(x-\mu)^2 dx \\ &= \int_2^7 0.2 \times (x-4.5)^2 dx \\ &= \frac{25}{12}\end{aligned}$$

Question 2

$$\begin{aligned}\mathrm{E}(X) &= \int_0^1 \left(x \times 3x^2\right) dx \\ &= \frac{3}{4}\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= \int_0^1 3x^2 \left(x - \frac{3}{4}\right)^2 \\ &= \frac{3}{80}\end{aligned}$$

Question 3

$$\begin{aligned}\mathrm{E}(X) &= \int_0^1 3x(x-1)^2 dx \\ &= 0.25\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(X) &= \int_0^1 3(x-1)^2 \left(x - \frac{1}{4}\right)^2 dx \\ &= 0.0375\end{aligned}$$

Question 4

$$\mathrm{E}(X) = \int_0^4 \frac{3x^{\frac{3}{2}}}{16} dx$$
$$= 2.4$$
$$\mathrm{Var}(X) = \int_0^4 \frac{3\sqrt{x}}{16} (x - 2.4)^2 dx$$
$$= \frac{192}{175}$$

Question 5

a

$$E(x) = \int_2^{12} \frac{x(12 - x)}{50} dx$$
$$= \frac{16}{3}$$

b

$$\mathrm{Var}(X) = \int_2^{12} \frac{(12 - x)}{50} \left(x - \frac{16}{3} \right)^2 dx$$
$$= \frac{50}{9}$$
$$\mathrm{SD}(X) = \sqrt{\frac{50}{9}} = \frac{5\sqrt{2}}{3}$$

Question 6

$$\mathrm{E}(X) = \int_0^{\infty} x\left(0.01e^{-0.01x}\right) dx$$
$$= 100 \text{ metres}$$

Question 7

$$\mathrm{E}(X) = \int_1^5 \frac{3x(6x - x^2 - 5)}{32}\,dx$$
$$= 3$$
$$\mathrm{Var}(X) = \int_1^5 \left(\frac{3(6x - x^2 - 5)}{32}(x-3)^2 \right) dx$$
$$= \frac{4}{5}$$
$$\mathrm{SD}(X) = \frac{2}{\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5} \sim 0.89$$

Question 8

$$\mathrm{E}(X) = 4.5 + 2 = 6.5$$
$$\mathrm{Var}(X) = \frac{25}{12}$$

Q1 pdf with a change of origin of 2 units.

Question 9

$$\mathrm{E}(X) = 4.5 \times 2 = 9$$
$$\mathrm{Var}(X) = \frac{25}{12} \times 2^2$$
$$= \frac{25}{3}$$

Q1 pdf with a change of scale $\times 2$

Question 10

a	$Y = 3X$	$\mathrm{E}(Y) = 12 \times 3 = 36$	$\mathrm{SD}(Y) = 3 \times 3 = 9$
b	$Y = X + 3$	$\mathrm{E}(Y) = 12 + 3 = 15$	$\mathrm{SD}(Y) = 3$
c	$Y = 2X + 5$	$\mathrm{E}(Y) = 2 \times 12 + 5 = 29$	$\mathrm{SD}(Y) = 3 \times 2 = 6$

Question 11

$$\begin{array}{lll} X: & \mu = 20 & \sigma = 4 \\ Z = 5X + 2: & \mu = 102 & \sigma = 20 \\ Z = 2X + 5: & \mu = 45 & \sigma = 8 \\ Z = 3X + 4: & \mu = 64 & \sigma = 12 \end{array}$$

a $Z = 5X + 2$ $\quad \mathrm{E}(Z) = 20 \times 5 + 2 = 102$ $\quad \mathrm{SD}(Z) = 4 \times 5 = 20$

b $Z = 2X + 5$ $\quad \mathrm{E}(Z) = 20 \times 2 + 5 = 45$ $\quad \mathrm{SD}(Z) = 4 \times 2 = 8$

c $Z = 3X + 4$ $\quad \mathrm{E}(Z) = 20 \times 3 + 4 = 64$ $\quad \mathrm{SD}(Z) = 4 \times 3 = 12$

Question 12

$X \quad \mathrm{E}(X) = 48 \quad \mathrm{Var}(X) = 16$

$Y = 1.8X + 32$

$\mathrm{E}(Y) = 48 \times 1.8 + 32 = 118.4$

$\mathrm{Var}(Y) = 16 \times 1.8^2 = 51.84$

$\mathrm{SD}(Y) = 4 \times 1.8 = 7.2$

Question 13

If random variable X involves lengths in centimetres and
random variable Y involves lengths in metres

$Y = X \div 100$

$\mathrm{E}(Y) = \mathrm{E}(X) \div 100$

$\mathrm{SD}(Y) = \mathrm{SD}(Y) \div 100$

Question 14

$$\int_0^k 0.25\,dx = \left[0.25x\right]_0^k$$
$$= 0.25k$$

$$\therefore \mathrm{P}(X \le x) = \begin{cases} 0 & x \le 0 \\ 0.25x & 0 < x \le 4 \\ 1 & x > 4 \end{cases}$$

Question 15

$$\int_2^k 0.25\,dx = \left[0.25x\right]_2^k$$
$$= 0.25k - 0.25(2)$$
$$= 0.25(k-2)$$

$$\therefore \text{P}(X \le x) = \begin{cases} 0 & x \le 2 \\ 0.25(x-2) & 2 < x \le 6 \\ 1 & x > 6 \end{cases}$$

Question 16

$$\int_0^k 3x^2 dx = \left[x^3\right]_0^k$$
$$= k^3$$
$$\therefore \text{P}(X < x) = \begin{cases} 0 & x \le 0 \\ x^3 & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

Question 17

$$\int_1^k \frac{1}{x} dx = \left[\ln x\right]_1^k$$
$$= \ln k - \ln 1$$
$$= \ln k$$
$$\therefore \text{P}(X \le x) = \begin{cases} 0 & x \le 1 \\ \ln x & 1 < x \le e \\ 1 & x > e \end{cases}$$

Question 18

$$\int_0^k e^{-x} dx = \left[-e^{-x}\right]_0^k$$
$$= -e^{-k} - \left(-e^{-0}\right)$$
$$= -e^{-k} + e^0$$
$$= 1 - e^{-k}$$
$$\therefore \text{P}(X \le x) = \begin{cases} 0 & x \le 0 \\ 1-e^{-x} & x > 0 \end{cases}$$

Question 19

$$\int_5^k (0.5-0.04x)\,dx = \left[0.5x-0.02x^2\right]_5^k$$

$$= 0.5k-0.02k^2-\left(2.5-\frac{1}{2}\right)$$

$$= 0.5k-0.02k^2-2$$

$$\therefore \mathrm{P}(X \le x) = \begin{cases} 0 & x \le 5 \\ 0.5x-0.02x^2-2 & 5 < x \le 10 \\ 1 & x > 10 \end{cases}$$

Question 20

a $\mathrm{P}(X \le 12) = 0.1(12-5)$

$= 0.7$

b $\mathrm{P}(X \le 8) = 0.1(8-5)$

$= 0.3$

c $\mathrm{P}(8 \le X \le 12) = 0.7-0.3$

$= 0.4$

d $\mathrm{P}(X > 8) = 1-\mathrm{P}(X \le 8)$

$= 0.7$

Question 21

a $\mathrm{P}(X \le 20)$

$= 1 - e^{-\frac{4}{3}}$

$= 0.7364$

b $\mathrm{P}(X \ge 20)$

$= 1 - \mathrm{P}(X \le 20)$

$= 0.2636$

c $\mathrm{P}(X \le 5)$

$= 1 - e^{-\frac{1}{3}}$

$= 0.2835$

d $\mathrm{P}(X \ge 20 \mid X \ge 15)$

$= \dfrac{\mathrm{P}(X \ge 20)}{\mathrm{P}(X \ge 15)}$

$= \dfrac{1-\left(1-e^{-\frac{4}{3}}\right)}{1-\left(1-e^{-1}\right)}$

$= \dfrac{e^{-\frac{4}{3}}}{e^{-1}}$

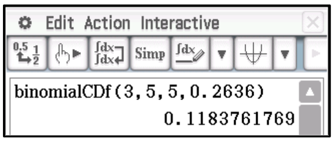
$= 0.7165$

e $\mathrm{P}(X \ge 20) = 0.2636$

$\mathrm{P}(\text{success}) = 0.2636$

$\binom{5}{3}(0.2636)^3(0.7364)^2 + \binom{5}{4}(0.2636)^4(0.7364) + \binom{5}{5}(0.2636)^5$

$= 0.1184$



Question 1

$$\begin{aligned} 3^x - 1 &= 5 \\ 3^x &= 6 \\ \log 3^x &= \log 6 \\ x \log 3 &= \log 6 \\ x &= \frac{\log 6}{\log 3} \end{aligned}$$

Question 2

a $\mathrm{P}(X \geq 0) = \frac{3}{5} = 0.6$

b $\mathrm{P}(1 \leq X \leq 2) = \frac{1}{5} = 0.2$

c

$$\begin{aligned} \mathrm{P}(X \leq 2 \mid X \geq 1) &= \frac{\mathrm{P}(1 \leq X \leq 2)}{\mathrm{P}(X \geq 1)} \\ &= \frac{0.2}{0.4} \\ &= \frac{1}{2} \end{aligned}$$

Question 3

a $\log_c 5$

$= \log_c\left(\frac{10}{2}\right)$

$= \log_c 10 - \log_c 2$

$= q - p$

b $\log_c 40$

$= \log_c(10 \times 2^2)$

$= \log_c 10 + \log_c 2^2$

$= \log_c 10 + 2\log_c 2$

$= 2p + q$

c $\log_c 200$

$= \log_c(10^2 \times 2)$

$= \log_c 10^2 + \log_c 2$

$= 2\log_c 10 + \log_c 2$

$= p + 2q$

d $\log_c(8c)$

$= \log_c 8 + \log_c c$

$= \log_c 2^3 + 1$

$= 3\log_c 2 + 1$

$= 3p + 1$

e $\log_2 10$

$= \frac{\log_c 10}{\log_c 2}$

$= \frac{q}{p}$

f $\log_{10} 2$

$= \frac{\log_c 2}{\log_c 10}$

$= \frac{p}{q}$

Question 4

a

$$e^x + e^{x+1} = 17$$
$$e^x(1+e) = 17$$
$$e^x = \frac{17}{e+1}$$
$$\ln e^x = \ln\left(\frac{17}{e+1}\right)$$
$$x \ln e = \ln\left(\frac{17}{e+1}\right)$$
$$x = \ln\left(\frac{17}{e+1}\right)$$

b

$$e^{2x+1} = 50^{x-7}$$
$$\ln e^{2x+1} = \ln 50^{x-7}$$
$$(2x+1)\ln e = (x-7)\ln 50$$
$$2x+1 = x\ln 50 - 7\ln 50$$
$$2x - x\ln 50 = -1 - 7\ln 50$$
$$x(2-\ln 50) = -(1+7\ln 50)$$
$$x = -\frac{(7\ln 50 + 1)}{(2-\ln 50)}$$
$$= \frac{7\ln 50 + 1}{\ln 50 - 2}$$

Question 5

a $\binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = 0.03215$

b $\left(\frac{1}{6}\right)^3 = 0.00463$

c $\binom{5}{4}\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)+\binom{5}{5}\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)^0$

$= \frac{5\times 5}{6^5}+\frac{1}{6^5}$

$= \frac{26}{6^5}$

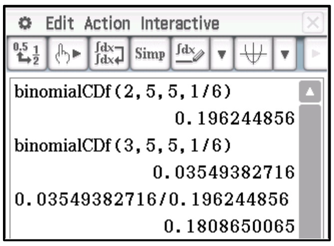
$= 0.00334$

d $\mathrm{P}(X>2\,|\,X>1)$

$= \frac{\mathrm{P}(X>3)}{\mathrm{P}(X>2)}$

$= \frac{0.035494}{0.196245}$

$= 0.18087$



Question 6

$\frac{d}{dx}(\ln 5x)$

$= \frac{5}{5x}$

$= \frac{1}{x}$

Question 7

$\frac{d}{dx}(3x+\ln 3x)$

$= 3+\frac{3}{3x}$

$= 3+\frac{1}{x}$

Question 8

$$\frac{d}{dx}(2\ln x)$$
$$=\frac{2}{x}$$

Question 9

$$\frac{d}{dx}\left(2\ln(x^3)\right)$$
$$=\frac{d}{dx}(6\ln x)$$
$$=\frac{6}{x}$$

Question 10

$$\frac{d}{dx}\left(\ln(2\sqrt{x})\right)$$
$$=\frac{2\times\frac{1}{2}\times x^{-\frac{1}{2}}}{2\sqrt{x}}$$
$$=\frac{1}{2\sqrt{x}\times\sqrt{x}}$$
$$=\frac{1}{2x}$$

Question 11

$$\frac{d}{dx}\left(\ln\left(\frac{2}{x}\right)\right)$$
$$=\frac{d}{dx}(\ln 2-\ln x)$$
$$=-\frac{1}{x}$$

Question 12

$$A = A_0 0.95^t$$
$$0.2A_0 = A_0 0.95^t$$
$$0.2 = 0.95^t$$
$$\log 0.2 = t \log 0.95$$
$$t = \frac{\log 0.2}{\log 0.95}$$
$$= 31.38$$

$\therefore$ Approximately 31 years.

Question 13

a

$$\frac{dy}{dx} = 1 + \frac{1}{x}$$
$$\frac{3}{2} = 1 + \frac{1}{x}$$
$$\frac{1}{x} = \frac{1}{2}$$
$$x = 2$$

When $x = 2$,

$$y = 2 + \ln(2 \times 2)$$
$$= 2 + \ln 4$$

$\therefore (2,\ 2 + \ln 4)$

b

$$y = \ln x + \ln(x+3)$$
$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x+3}$$
$$\frac{1}{2} = \frac{x + x + 3}{x(x+3)}$$
$$4x + 6 = x^2 + 3x$$

$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = -2,\ 3 \quad (x > 0)$$
$$x = 3$$

When $x = 3$,

$$x(x+3) = 18$$
$$y = \ln 18$$

$\therefore (3,\ \ln 18)$

Question 14

a $$\frac{dC}{dx} = 200 \times \frac{1}{x+1}$$
$$= \frac{200}{x+1} \text{ \$/unit}$$

b $$\frac{200}{x+1} = 2$$
$$x+1 = 100$$
$$x = 99$$
$$C(99) = 600 + 200\ln(100)$$
$$\therefore \text{Average cost}: \frac{600 + 200\ln(100)}{99}$$
$$= \$15.36$$

Question 15

$$y = \ln 2 + \ln \sin x$$
$$\frac{dy}{dx} = \frac{\cos x}{\sin x}$$
$$= \frac{\cos x}{\sin x}$$
When $x = \frac{\pi}{6}$,
$$\frac{dy}{dx} = \frac{\cos\frac{\pi}{6}}{\sin\frac{\pi}{6}}$$
$$= \sqrt{3}$$

Equation of tangent
$$y = \sqrt{3}x + c$$
Using $\left(\frac{\pi}{6}, 0\right)$
$$0 = \sqrt{3} \times \frac{\pi}{6} + c$$
$$c = -\frac{\sqrt{3}\pi}{6}$$
$$\therefore y = \sqrt{3}x - \frac{\sqrt{3}\pi}{6}$$

Question 16

a $P(X \le 8) = 1 - e^{-\frac{8}{8}}$

$= 0.6321$

b $P(X \le 24) = 1 - e^{-3}$

$= 0.9502$

c $1 - 0.9502 = 0.0498$

Question 17

Stationary points when $\frac{dy}{dx} = 0$.

$$\frac{dy}{dx} = 4x - \frac{1}{x}$$

$$0 = 4x - \frac{1}{x}$$

$$\frac{1}{x} = 4x$$

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \frac{1}{2} \quad (x > 0)$$

When $x = \frac{1}{2}$,

$$y = 2\left(\frac{1}{2}\right)^2 - \log_e\left(\frac{1}{2}\right)$$

$$= \frac{1}{2} - \log_e 2^{-1}$$

$$= \frac{1}{2} + \log_e 2$$

$$\therefore \left(\frac{1}{2}, \frac{1}{2} + \log_e 2\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(4x - \frac{1}{x}\right)$$

$$= 4 - (-1)x^{-2}$$

$$= 4 + \frac{1}{x^2}$$

When $x = \frac{1}{2}$,

$$\frac{d^2y}{dx^2} = 4 + \frac{1}{\left(\frac{1}{2}\right)^2}$$

$$= 8$$

$\therefore$ As $\frac{d^2y}{dx^2} > 0, \left(\frac{1}{2}, \frac{1}{2} + \log_e 2\right)$ is a minimum point.

Question 18

$$\int_0^\infty ae^{-bx}dx = -\frac{a}{b}\int_0^\infty \left(-be^{-bx}\right)dx$$

$$= -\frac{a}{b}\left[e^{-bx}\right]_0^\infty$$

$$1 = -\frac{a}{b}(e^{-\infty} - e^0)$$

$$-\frac{b}{a} = \frac{1}{e^\infty} - 1$$

As $x \to \infty, \frac{1}{e^\infty} \to 0$

$\Rightarrow -\frac{b}{a} = -1$

$\Rightarrow b = a$

$$\mathrm{E}(X) = \int_0^\infty 0.25xe^{-0.25x}dx$$

$$= 4$$

$$\mathrm{Var}(X) = \int_0^\infty 0.25e^{-0.25x}(x-4)^2\,dx$$

$$= 16$$